Geometric Flows of Polygons

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Abstract

We study polygon flows, in which the vertices of a polygon move according to a differential equation. Previously Chow and Glickenstein [1] studied a linear flow that takes polygons to affine transformations of regular polygons. We study two nonlinear flows, which we conjecture take polygons to regular polygons. We present numerical evidence for our conjecture and prove the conjecture in certain symmetry classes.

Polygon Flows

We describe a polygon by an ordered list of vectors $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ in $\mathbb{R}^2$, which correspond to the vertices of the polygon.

A vector field $\mathbf{V} = \{V_1, \ldots, V_n\}$ on polygon $\mathbf{X}$ is a list of vectors , where we consider the vector V_k to be located at vertex X_k. Two important vector fields are $\mathbf{L}$ and $\mathbf{R}$, defined by

$$L_k = X_{k-1} - X_k \quad \text{and} \quad R_k = X_{k+1} - X_k.$$

The flows we consider take the form

$$\frac{d}{dt}\mathbf{X} = \mathbf{V},$$

by which we mean $\frac{d}{dt}X_k = V_k$.

The flow considered by Chow and Glickenstein is

$$\mathbf{V}_{CG} = \mathbf{L} + \mathbf{R}$$

Our Flows

Under the flow determined by $\mathbf{V}_{CG}$, polygons collapse to points; at the point of collapse they become affine transformations of regular polygons. We seek flows where polygons collapse to regular polygons. The flows we consider are:

Squared Chow-Glickenstein Flow

$$\mathbf{V}_{SCG} = |\mathbf{L} + \mathbf{R}|(\mathbf{L} + \mathbf{R})$$

Modified Curvature Flow

$$\mathbf{V}_{M\kappa} = \frac{1}{|\mathbf{L} - \mathbf{R}|^2}(\mathbf{L} + \mathbf{R}).$$

The Modified Curvature Flow

The Modified Curvature Flow is a simplification of the Menger Curvature Flow studied by Jecko and Léger [2]. To understand $\mathbf{V}_{M\kappa}$, consider Figure 1. The top vertex moves faster, relative to the motion of the right vertex, than under $\mathbf{V}_{CG}$. The Modified Curvature Flow collapses diamonds to squares.

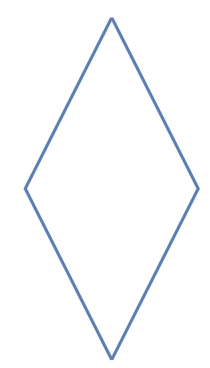

Figure 1: The diamond, being an affine transformation of a square, collapses self similarly under $\mathbf{V}_{CG}$. Under both $\mathbf{V}_{SCG}$ and $\mathbf{V}_{M\kappa}$ the diamond collapse to a square.

Squared Chow-Glickenstein Flow

This flow modifies $\mathbf{V}_{CG}$ so that the relative velocities of the vertices are increased. Consider the diamond in Figure 1, the $\mathbf{V}_{SCG}$ flow accentuates the relative velocities of the top and right vertices so that the polygon collapses to a square.

Conjecture

We conjecture that under either of our two flows all polygons, except rectangles, become regular polygons at the time of collapse.

Our conjecture is supported by numerical evidence.

Rectangles, which can be inscribed on a circle, flow self similarily under all three flows discussed here.

Results

We have proven our conjecture in three different symmetry classes.

Isoperimetric Ratio

For polygon $\mathbf{X}$ with n vertices, denote the perimeter by $\mathcal{L}$ and the enclosed area by $\mathcal{A}$. All simple polygons satisfy

$$\frac{\mathcal{L}^2}{\mathcal{A}} \geq 4n \tan\left(\frac{\pi}{n}\right),$$

with equality if and only if the polygon is regular. The isoperimetric ratio on the left measure the extent to which the polygon is regular.

Our numerical simulations show that the isoperimetric ratio decreases towards the optimal value under our flows. Consider, for example, the polygon appearing in Figure 2. The isoperimetric ratio for the evolution of this polygon under our two flows is shown in Figures 3 and 4.

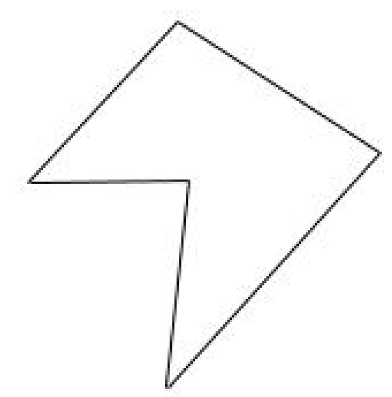

Figure 2: An example Polygon

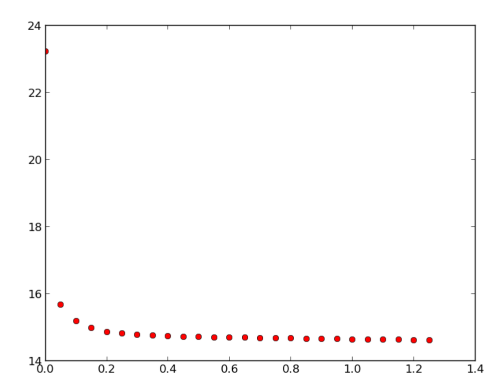

Figure 3: The Isoperimetric ratio for the RSCG flow

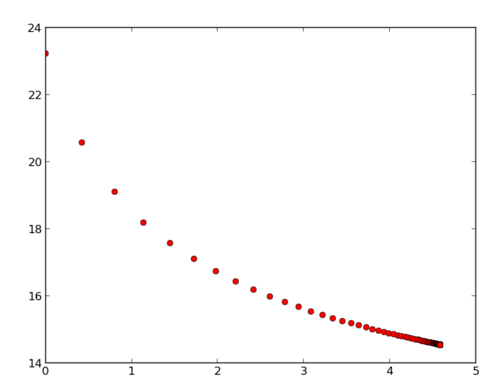

Figure 4: The isoperimetric ratio of the modified curvature flow

Symmetric Cases

We consider flow of polygons with maximal reflection symmetry. Such symmetry is preserved by all three flows. Under this symmetry assumption, the differential equations describing our flows reduce to a system with two unknowns.

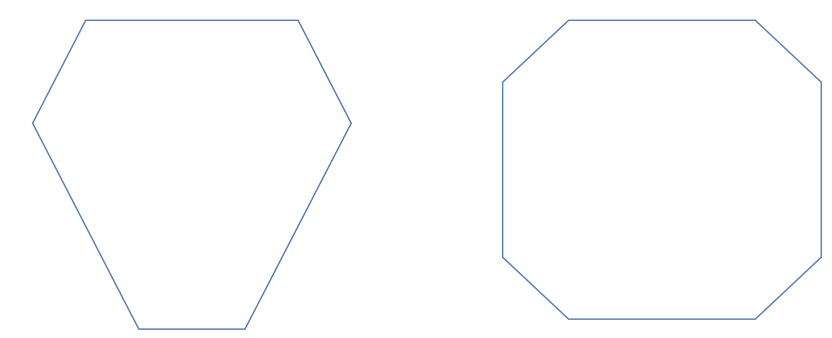

Figure 5: Typical hexagons and octagons in the symmetry classes we study. We have proven that such polygons collapse to regular polygons under our flows.

In this symmetry setting we show by direct computation that the isoperimetric ratio decreases under our flows. We then show that polygons do not collapse without the isoperimetric ratio achieving the optimal value.

References

[1] Bennett Chow and David Glickenstein. Semidiscrete geometric flows of polygons. *Amer. Math. Monthly*, 114(4):316–328, 2007.

[2] Thierry Jecko and Jean-Christophe Léger. Polygon shortening makes (most) quadrilaterals circular. *Bull. Korean Math. Soc.*, 39(1):97–111, 2002.

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